Trigonometry Workbook Solutions

The law of sines and law of cosines

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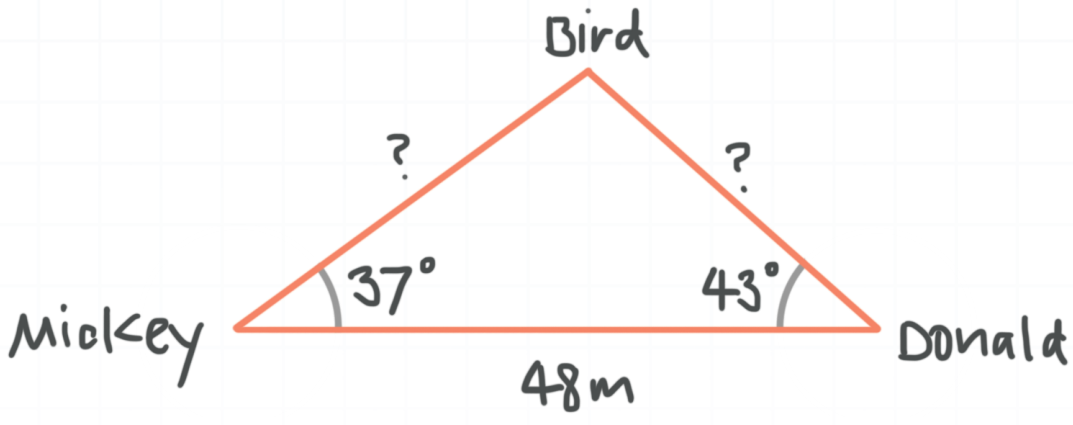
MATH

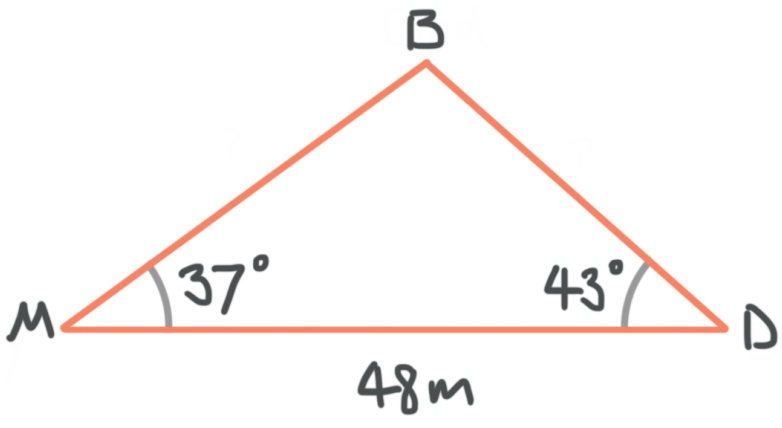
LAW OF SINES

■ 1. Mickey and Donald stand on different sides of a tree. Each of them sees the same bird in the tree. They measure the angles of elevation from themselves to the bird, and get $37°$ and $43°$ respectively. If Mickey and Donald are 48 m apart, find the distances from Mickey and Donald to the bird, rounded to the nearest meter.



Solution:

If M is Mickey, B is the bird, and D is Donald, then we have the triangle MBD.



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In the triangle, $m\angle M = 37°$, $m\angle D = 43°$, the length of MD is 48 m, and MB and DB are the sides we're interested in.

Since the sum of the interior angles of any triangle is $180°$, the measure of angle B is:

$$180° - 37° - 43° = 100°$$

Since we know all of the angles and one side, we can apply the law of sines for triangle MBD:

$$\frac{MD}{\sin B} = \frac{MB}{\sin D} = \frac{DB}{\sin M}$$

$$\frac{48}{\sin 100°} = \frac{MB}{\sin 43°} = \frac{DB}{\sin 37°}$$

Focus on the left half of this equation.

$$\frac{48}{\sin 100°} = \frac{MB}{\sin 43°}$$

$$MB = \frac{48 \sin 43°}{\sin 100°} \approx 33 \text{ m}$$

Then DB is

$$DB = \frac{48 \sin 37°}{\sin 100°} \approx 29 \text{ m}$$

■ 2. In triangle ABC, find the ratio of AC to AB, given that $m\angle A = \pi/6$ and $m\angle B = 2\pi/3$.

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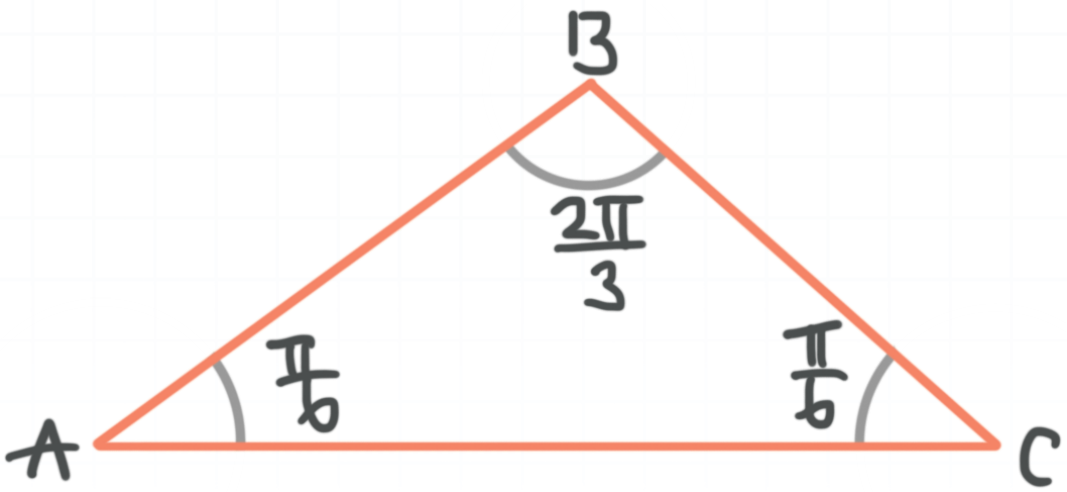
Solution:

Since the sum of the interior angles of any triangle is π, the measure of angle C is

$$\pi - \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

So the triangle looks like this:



Write down the law of sines for triangle ABC.

$$\frac{AB}{\sin C} = \frac{BC}{\sin A} = \frac{AC}{\sin B}$$

From this, pull out the left and right sides.

$$\frac{AB}{\sin C} = \frac{AC}{\sin B}$$

$$AB \cdot \frac{\sin B}{\sin C} = AC$$

$$\frac{\sin B}{\sin C} = \frac{AC}{AB}$$

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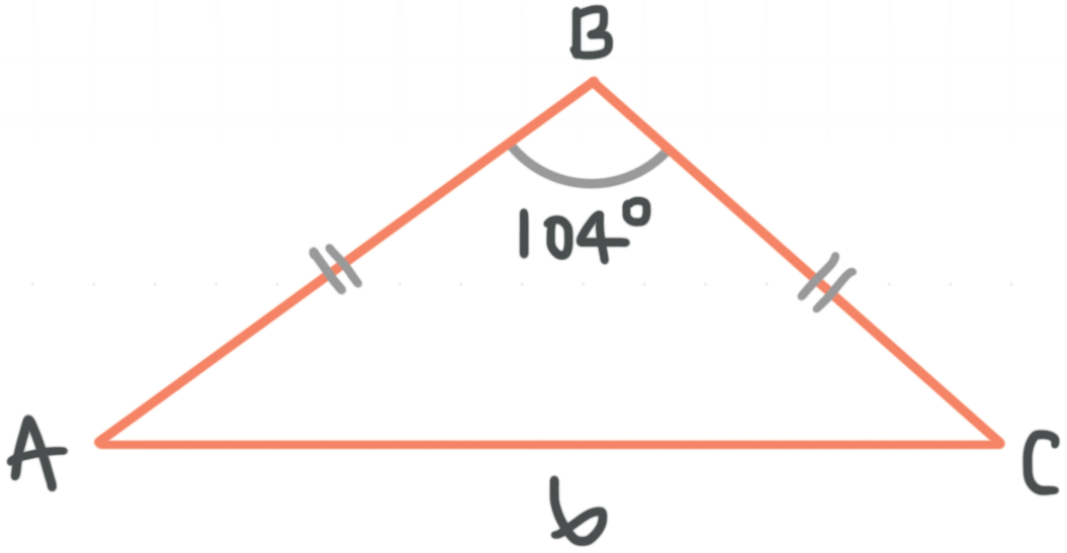
Plug in the sine values.

$$\frac{\sin\frac{2\pi}{3}}{\sin\frac{\pi}{6}} = \frac{AC}{AB}$$

$$\frac{AC}{AB} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

So the ratio of AC to AB is $\sqrt{3}$.

■ 3. The base of an isosceles triangle is 6 units long, and its opposite angle is $104°$. Find the triangle's perimeter.



Solution:

Since the sum of the interior angles of any triangle is $180°$,

$$A + B + C = 180°$$

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Since the angles adjacent to the base in the isosceles triangle are equal, we can plug in $C = A$ and $B = 104°$, so

$$2A + 104° = 180°$$

$$2A = 76°$$

$$A = 38°$$

Then $C = 38°$ as well. So use the law of sines to find AB.

$$\frac{AB}{\sin C} = \frac{BC}{\sin A} = \frac{AC}{\sin B}$$

Take the left and right sides from this equation

$$\frac{AB}{\sin C} = \frac{AC}{\sin B}$$

$$AB = \frac{AC \cdot \sin C}{\sin B}$$

$$AB = \frac{6 \cdot \sin 38°}{\sin 104°} \approx 3.8$$

Then the perimeter is the sum of all three sides.

$$AB + BC + AC$$

$$3.8 + 3.8 + 6 = 13.6$$

$$13.6$$

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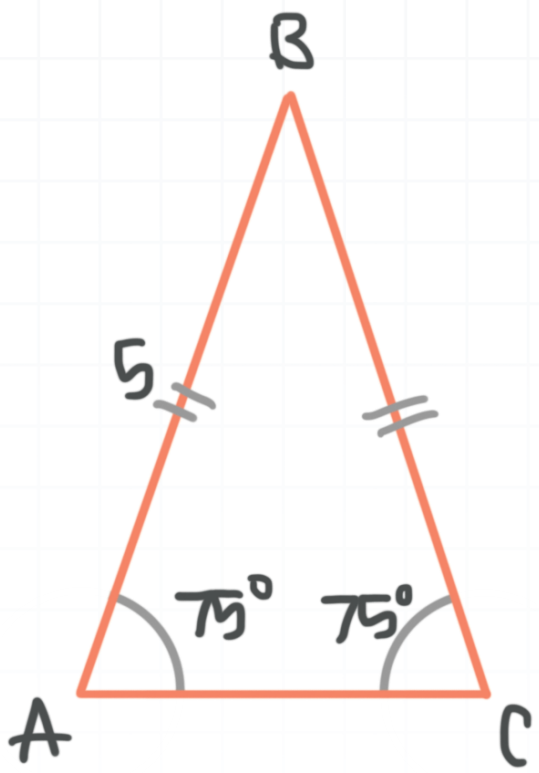
LAW OF COSINES

■ 1. Find the perimeter of the isosceles.



Solution:

ABC is an isosceles triangle with sides $AB = BC = 5$, and angles $A = C = 75$. In order to find the perimeter, we need to know the length of the base. But before applying the law of cosines, we need to find the angle B between the legs.

Since the sum of the interior angles of any triangle is $180°$,

$$A + B + C = 180°$$

$$75° + B + 75° = 180°$$

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$$B = 180^\circ - 75^\circ - 75^\circ$$

$$B = 30^\circ$$

Apply the law of cosines.

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos B$$

$$AC = \sqrt{AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos B}$$

$$AC = \sqrt{5^2 + 5^2 - 2 \cdot 5 \cdot 5 \cdot \cos 30^\circ}$$

$$AC \approx 2.6$$

So the perimeter of the triangle is

$$AB + BC + AC$$

$$5 + 5 + 2.6$$

$$12.6$$

■ 2. Bob rides a Ferris wheel at a carnival. Find the distance from Bob to the lowest point of the wheel, given that the diameter of the wheel is 24 m, and that the angle of elevation of Bob with respect to the center of the wheel is 50°.

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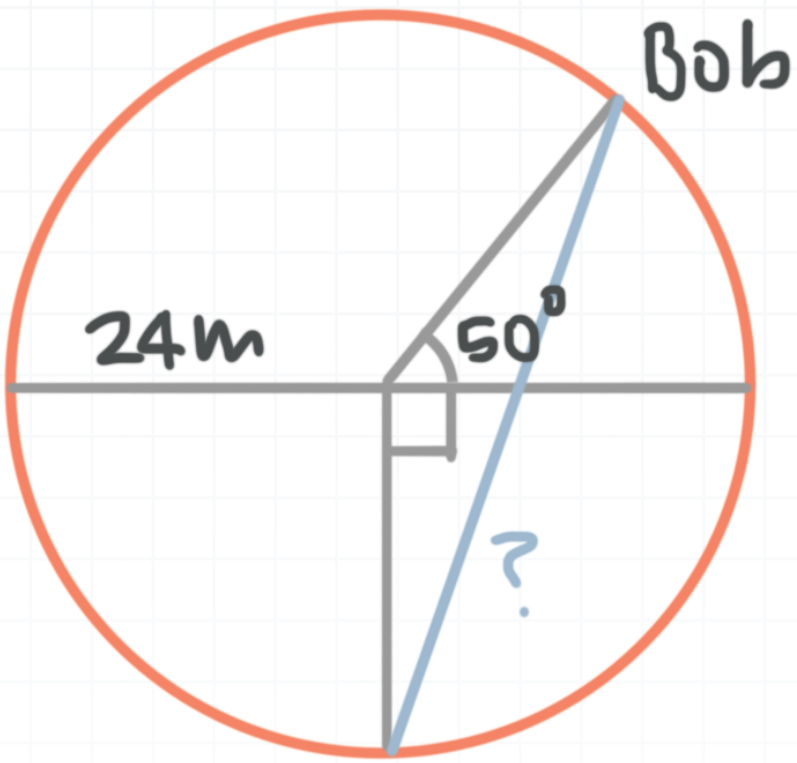
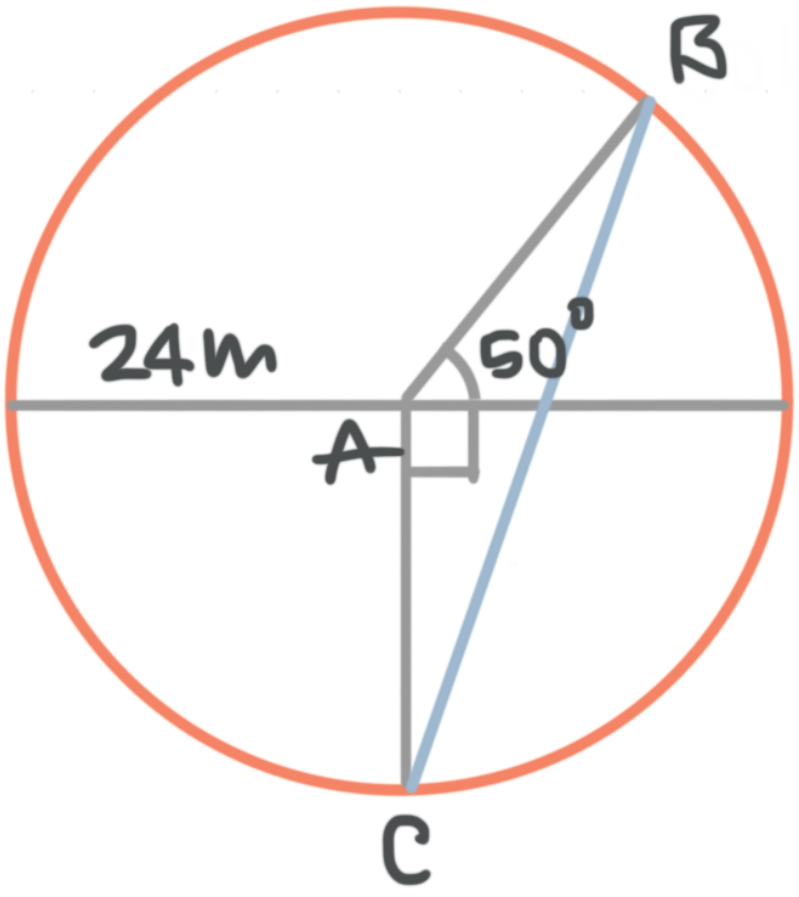


Solution:

Think about the problem as the triangle ABC, where A is the center of the wheel, B is where Bob is, and C is the lowest point of the wheel.



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In the triangle, AB and AC are the radii of the wheel. Since the radius is the half the diameter,

$$AB = AC = \frac{24 \text{ m}}{2} = 12 \text{ m}$$

The angle A between the radii is equal to the angle of elevation plus $90°$.

$$A = 90° + 50°$$

$$A = 140°$$

Since we know two adjacent sides in the triangle, and the angle between them, we can apply the law of cosines.

$$BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos A$$

$$BC = \sqrt{AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos A}$$

$$BC = \sqrt{12^2 + 12^2 - 2 \cdot 12 \cdot 12 \cdot \cos 140°}$$

$$BC \approx 23 \text{ m}$$

■ 3. In triangle ABC where $A(2,3)$, $B(3, -1)$, and $C(-3, -2)$, find the cosine of angle A.

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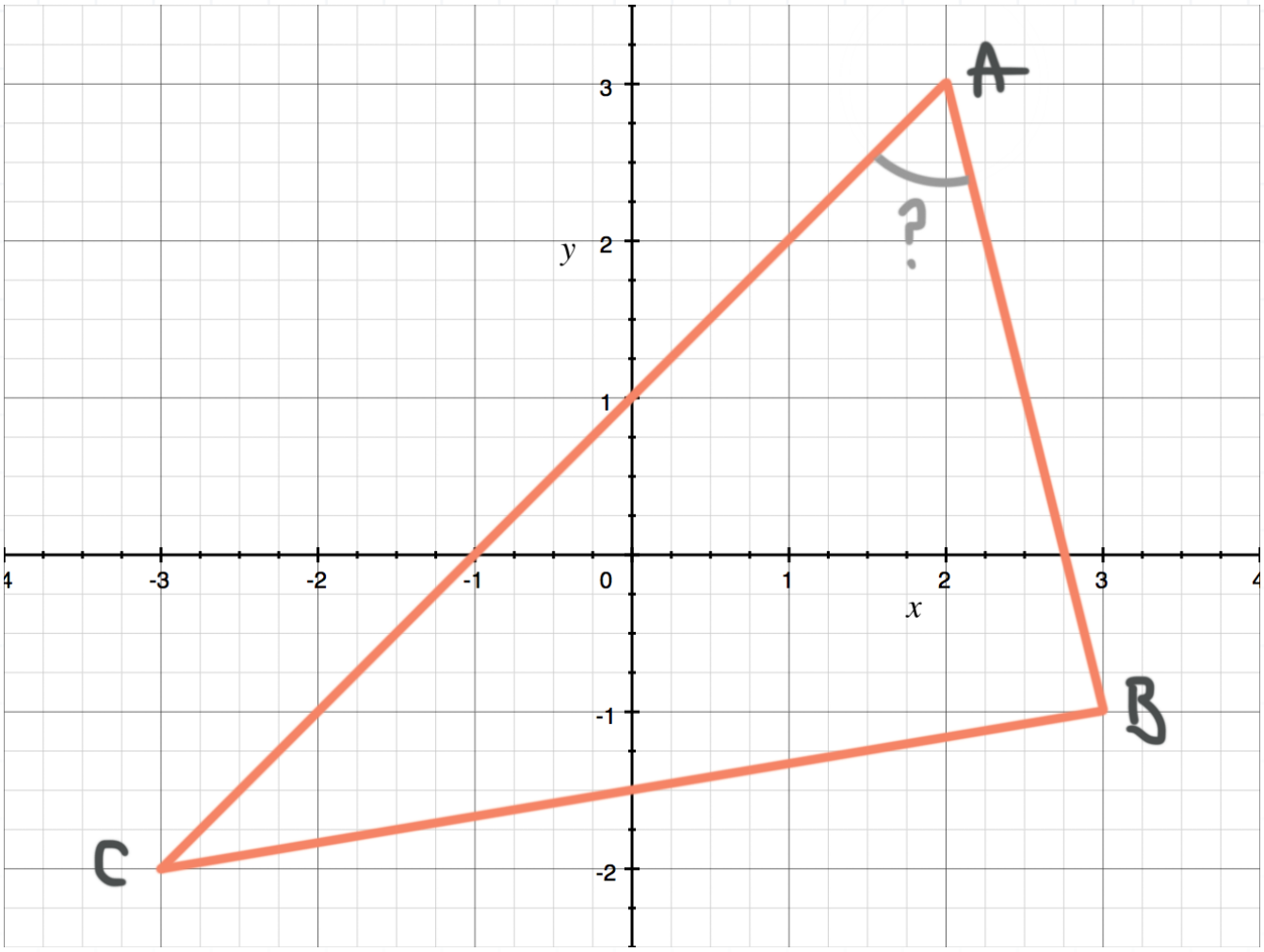

Solution:

We can find the cosine of the angle A using the law of cosines, but before using it, we need to find the length of all three sides. Use the Pythagorean theorem and helper rectangle $CDEF$ to find them.

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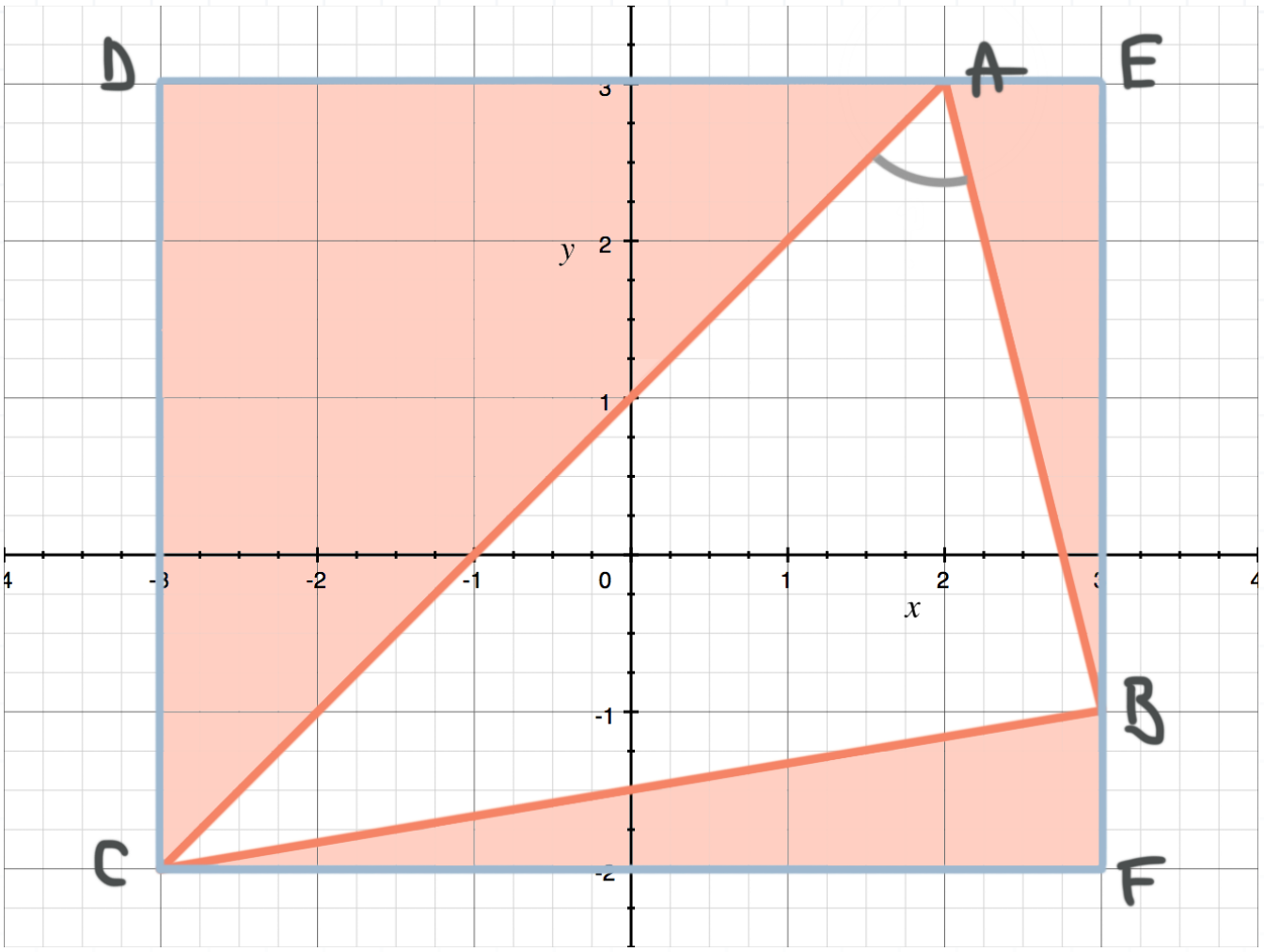

So the triangle's side lengths are

$$AB^2 = AE^2 + EB^2 = 1^2 + 4^2 = 17 \qquad AB = \sqrt{17}$$

$$BC^2 = BF^2 + FC^2 = 1^2 + 6^2 = 37 \qquad BC = \sqrt{37}$$

$$AC^2 = CD^2 + DA^2 = 5^2 + 5^2 = 50 \qquad AC = \sqrt{50}$$

Use the law of cosines for triangle ABC to find angle A.

$$\cos A = \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC}$$

$$\cos A = \frac{17 + 50 - 37}{2 \cdot \sqrt{17} \cdot \sqrt{50}} \approx 0.51$$

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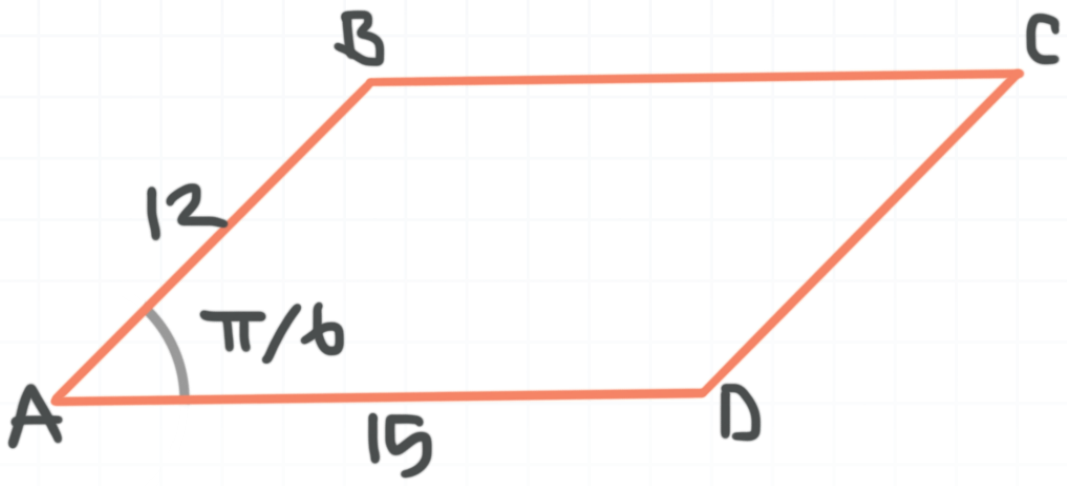
LAW OF SINES FOR AREA OF A TRIANGLE

1. Find the area of parallelogram $ABCD$ given that $AB = 12$, $AD = 15$, and $m\angle A = \pi/6$.



Solution:

Parallelogram $ABCD$ consists of two equal triangles, ABD and BCD. Since equal triangles have equal areas,

$$\text{Area}_{ABCD} = 2 \cdot \text{Area}_{ABD}$$

For triangle ABD, we know two adjacent sides and the angle between them, so we can apply the law of sines for area of a triangle.

$$\text{Area}_{ABCD} = 2 \cdot \text{Area}_{ABD} = 2 \cdot \frac{1}{2} \cdot AB \cdot AD \cdot \sin A$$

$$\text{Area}_{ABCD} = AB \cdot AD \cdot \sin A$$

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The area of any parallelogram with two adjacent sides a and b and their included angle θ is $ab \sin\theta$.

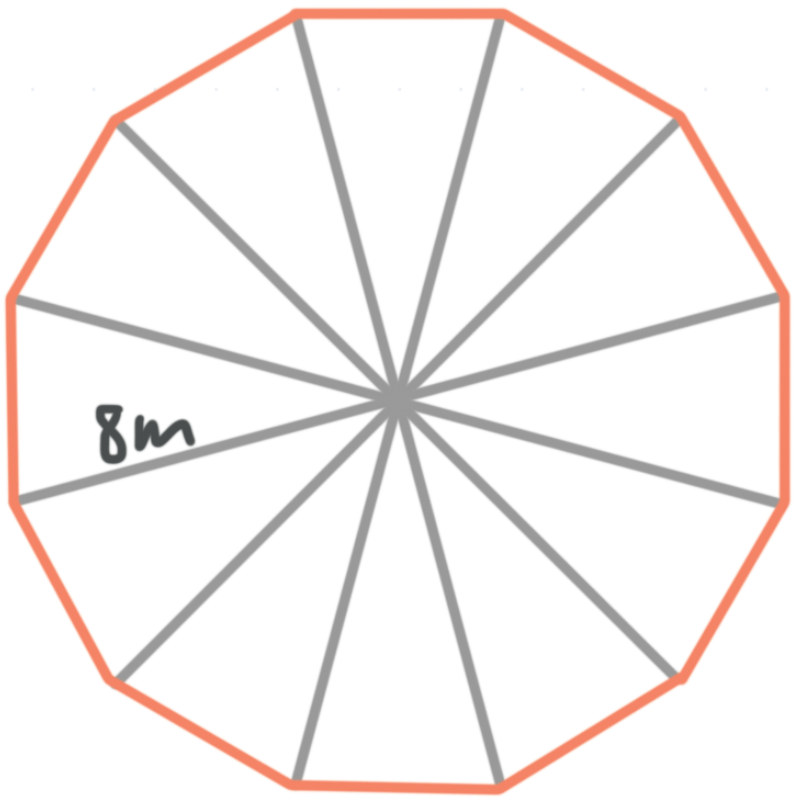
$$A = ab \sin\theta$$

$$A = 12 \cdot 15 \cdot \sin\frac{\pi}{6}$$

$$A = 12 \cdot 15 \cdot \frac{1}{2}$$

$$A = 90$$

■ 2. Maggie rides a Ferris wheel at a carnival. There are 12 cabins in the wheel, each positioned at a distance of 8 m from the center of the wheel. So the 12 cabins form a dodecagon (twelve-sided polygon). Find the area of the Ferris wheel.



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Solution:

Since the dodecagon consists of the twelve equal triangles, we can find the area of one triangle, and then multiply that area by 12.

For each triangle, we know that the two adjacent sides that touch the center of the wheel are 8 m. In order to find the angle between these sides, let's use the fact that 12 such angles form the complete angle of $360°$, so each angle is equal to

$$\frac{360°}{12} = 30°$$

Now, we can apply the law of sines for area of a triangle.

$$A_{\text{triangle}} = \frac{1}{2} \cdot 8 \cdot 8 \cdot \sin 30°$$

$$A_{\text{triangle}} = \frac{1}{2} \cdot 8 \cdot 8 \cdot \frac{1}{2}$$

$$A_{\text{triangle}} = 16 \text{ m}^2$$

Find the area of the ferris wheel by multiplying this area by 12.

$$A_{\text{dodecagon}} = 12 \cdot A_{\text{triangle}}$$

$$A_{\text{dodecagon}} = 12 \cdot 16$$

$$A_{\text{dodecagon}} = 192 \text{ m}^2$$

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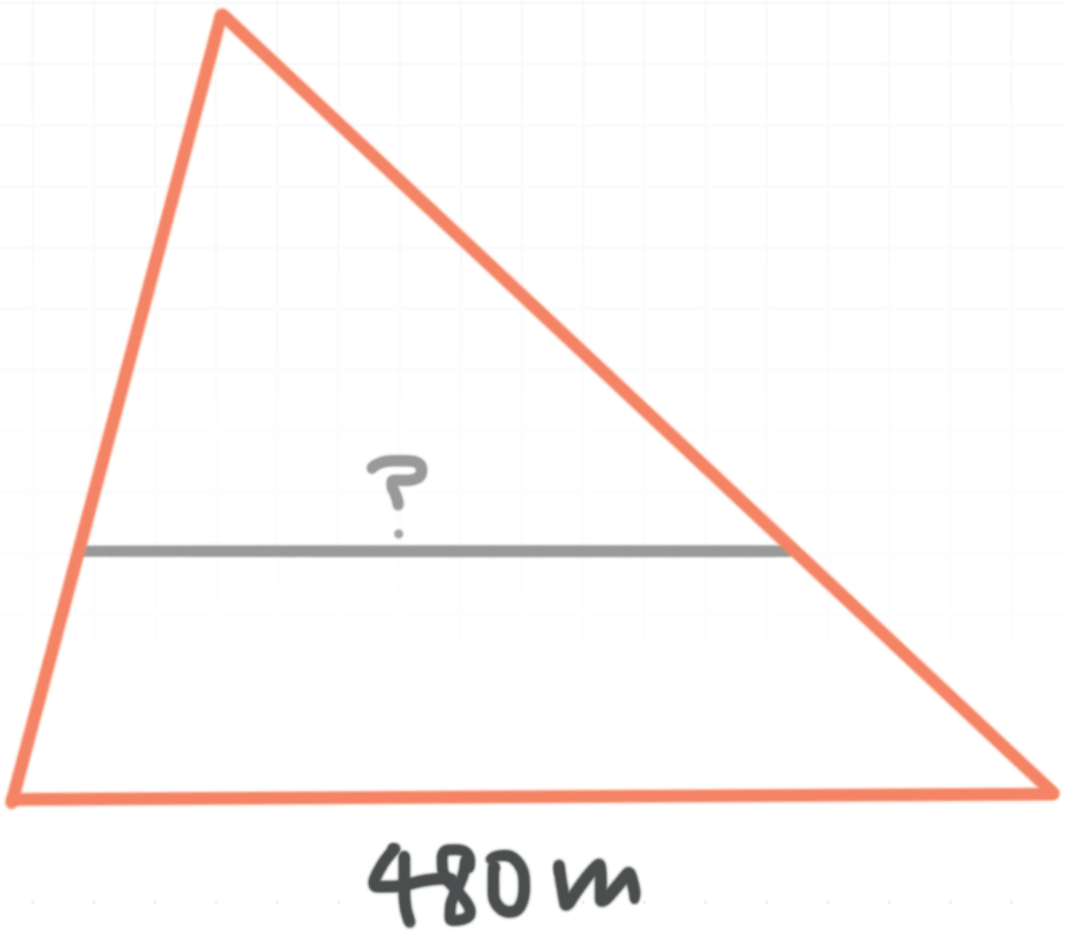
■ 3. A triangular-shaped plot of land has a base length 480 m that sits along a river. The land owner decides to split the plot into two equal parts (in terms of area), using a straight fence parallel to the river. Find the length of the fence.



Solution:

Call the triangle ABC with the base $AC = 480$ m. Let A_1 be the point on the side AB, and C_1 the point on the side BC, such that AC is parallel to A_1C_1, and the line A_1C_1 divides the triangle into two equal parts in terms of area.

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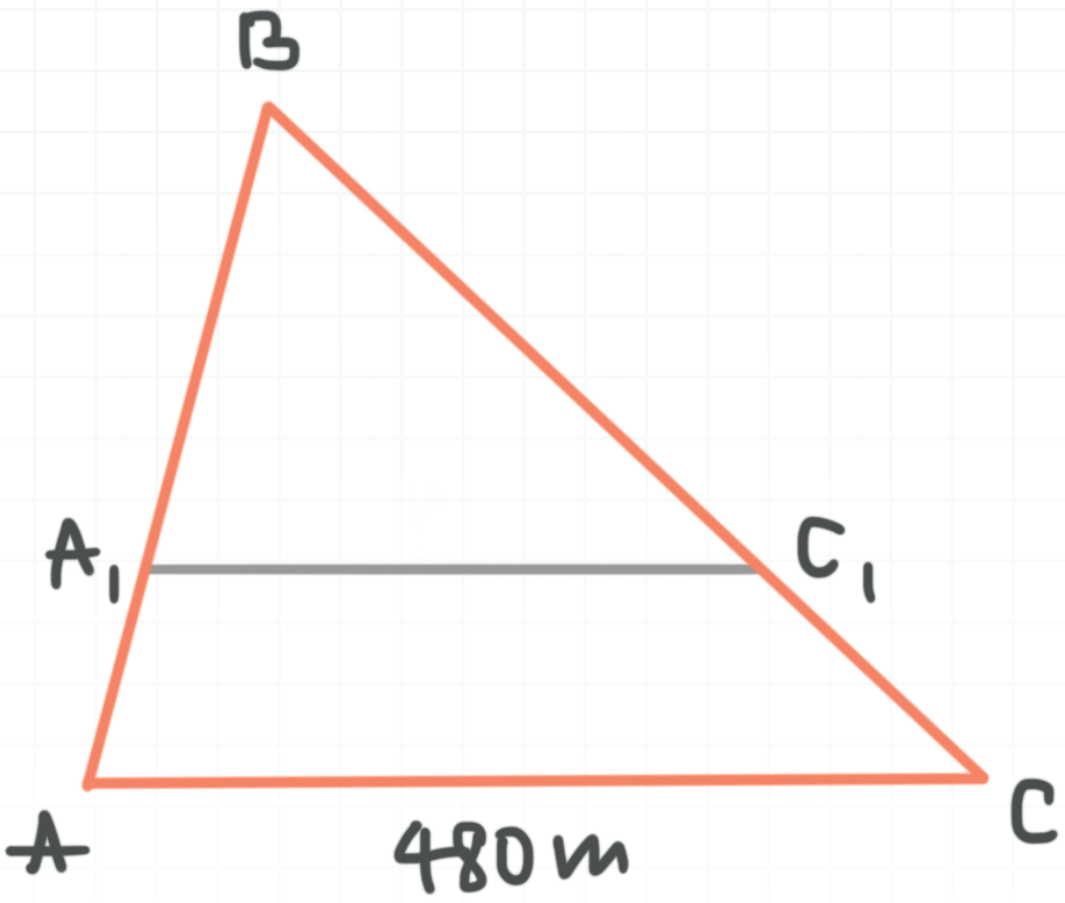

The triangles ABC and A_1BC_1 are similar, so their sides are proportional with some coefficient k.

$$\frac{AB}{A_1B} = \frac{BC}{BC_1} = \frac{AC}{A_1C_1} = k$$

Apply the law of sines for area of a triangle to the similar triangles.

$$A_{ABC} = \frac{1}{2} \cdot AB \cdot BC \cdot \sin B$$

$$A_{A_1BC_1} = \frac{1}{2} \cdot A_1B \cdot BC_1 \cdot \sin B$$

Then divide the first equation by the second.

$$\frac{A_{ABC}}{A_{A_1BC_1}} = \frac{\frac{1}{2} \cdot AB \cdot BC \cdot \sin B}{\frac{1}{2} \cdot A_1B \cdot BC_1 \cdot \sin B}$$

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$$\frac{A_{ABC}}{A_{A_1BC_1}} = \frac{AB}{A_1B} \cdot \frac{BC}{BC_1}$$

We know that the area of ABC is twice as big as the area of A_1BC_1, so

$$2 = \frac{AB}{A_1B} \cdot \frac{BC}{BC_1}$$

Because

$$\frac{AB}{A_1B} = \frac{BC}{BC_1} = k$$

we get

$$2 = k^2$$

$$k = \sqrt{2}$$

Plugging this value for k back into

$$\frac{AB}{A_1B} = \frac{BC}{BC_1} = \frac{AC}{A_1C_1} = k$$

gives us the length of A_1C_1.

$$\frac{AC}{A_1C_1} = \sqrt{2}$$

$$A_1C_1 = \frac{AC}{\sqrt{2}} = \frac{480}{\sqrt{2}} = 240\sqrt{2} \approx 339.4 \text{ m}$$

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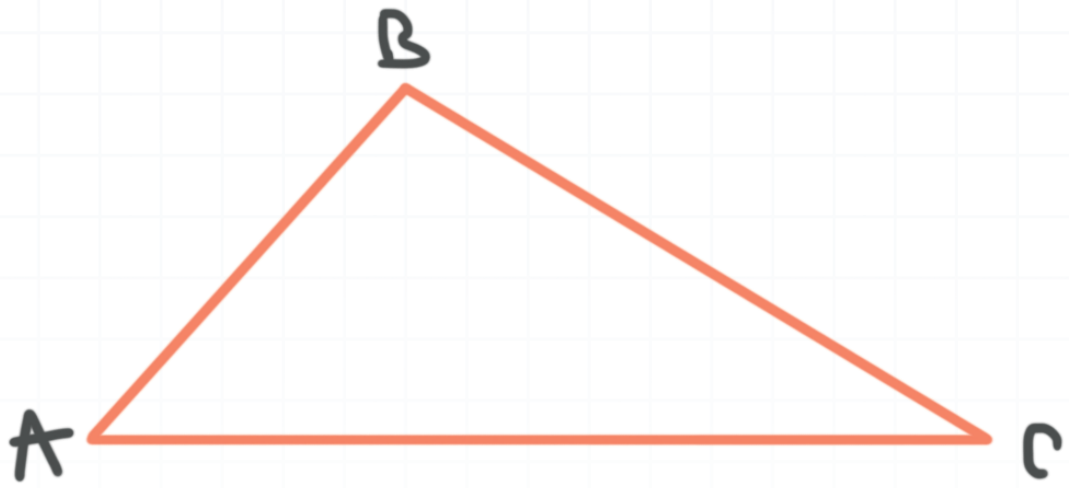
HERON'S FORMULA FOR AREA OF A TRIANGLE

■ 1. In triangle ABC, the length of three sides AB, BC, and AC are in a ratio of $2 : 3 : 4$. Find the area of ABC if its perimeter is 18.



Solution:

The ratio $2 : 3 : 4$ means that there exists some number x such that

$$AB = 2x$$

$$BC = 3x$$

$$AC = 4x$$

Since the sum of the side lengths is 18, we get.

$$AB + BC + AC = 18$$

$$2x + 3x + 4x = 18$$

$$9x = 18$$

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$$x = 2$$

Then the side lengths are

$$AB = 2(2) = 4$$

$$BC = 3(2) = 6$$

$$AC = 4(2) = 8$$

Apply Heron's formula for triangle ABC. Since the perimeter of the triangle is 18, the semi-perimeter is $18/2 = 9$. So by Heron's formula,

$$A = \sqrt{s(s - AB)(s - BC)(s - AC)}$$

$$A = \sqrt{9(9 - 4)(9 - 6)(9 - 8)}$$

$$A = \sqrt{135}$$

$$A = 3\sqrt{15}$$

■ 2. Use Heron's formula to find the area of triangle ABC, where $A(-1,4)$, $B(3,2)$, and $C(1, -1)$.

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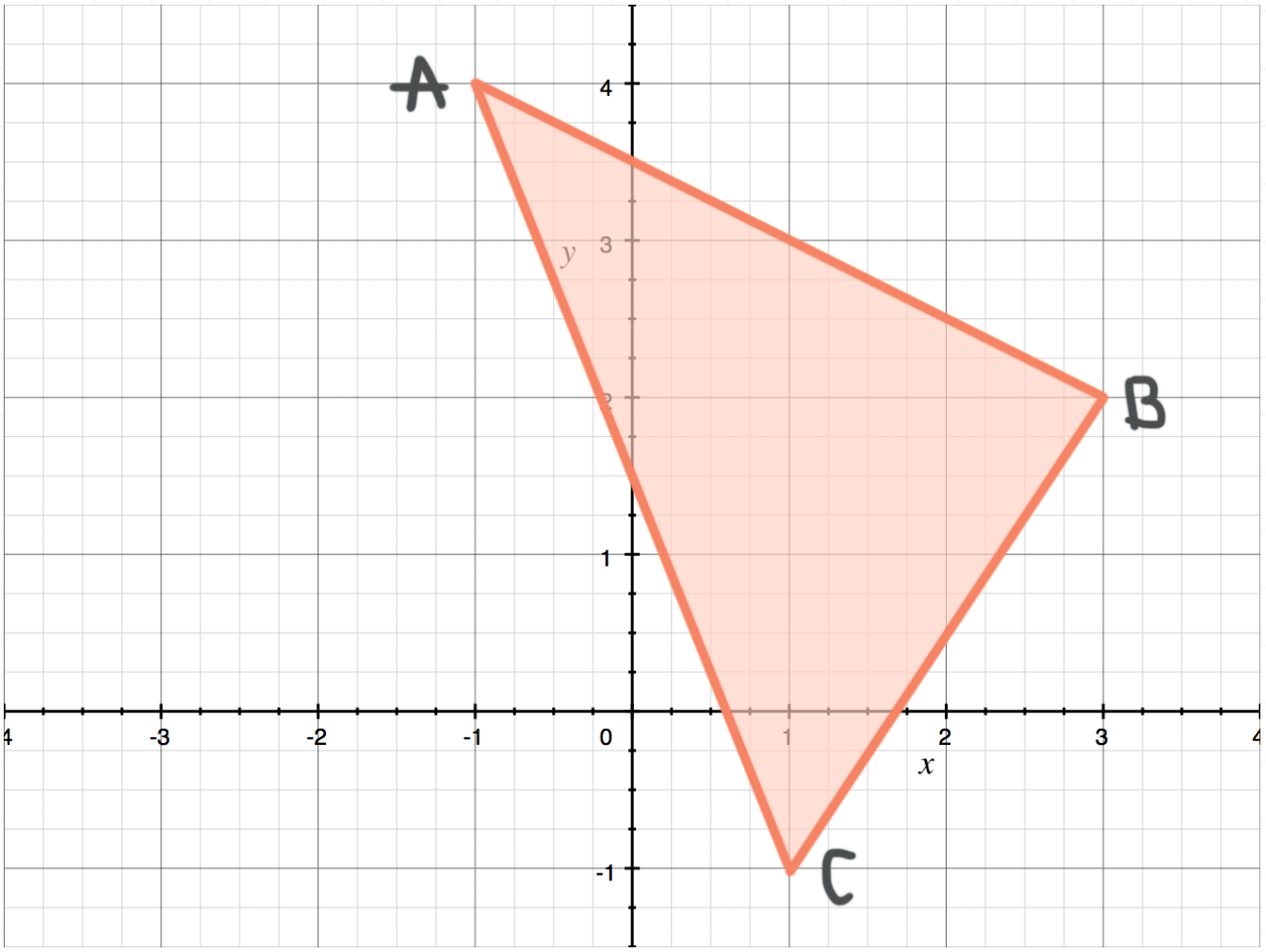

Solution:

In order to apply Heron's formula, we need to know the length of all three sides of the triangle. Let's use Pythagorean theorem and helper rectangle $ADEF$ to find them.

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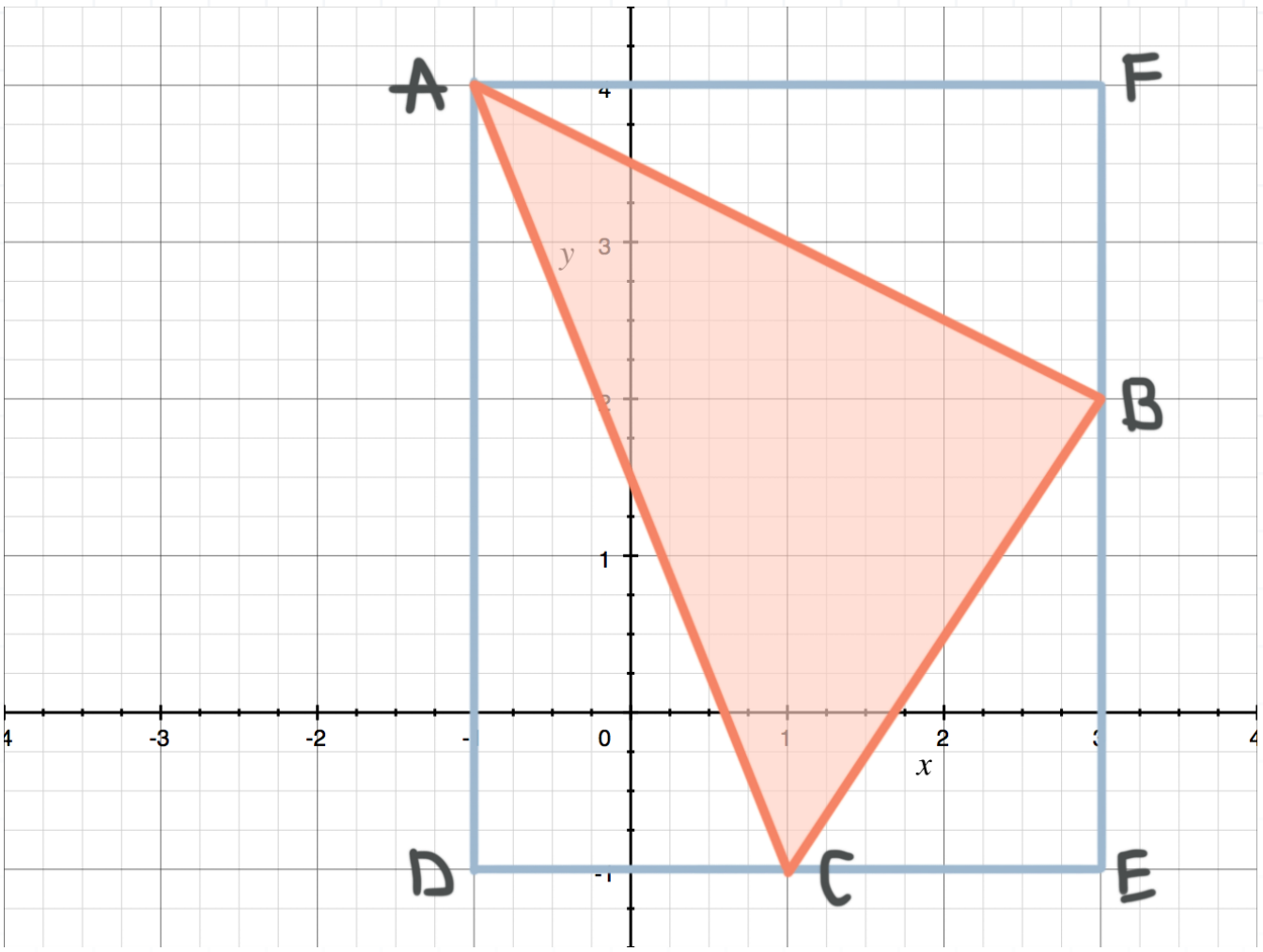

The three side lengths are

$$AB^2 = AF^2 + FB^2 = 4^2 + 2^2 = 20 \qquad AB = \sqrt{20} \approx 4.47$$

$$BC^2 = BE^2 + EC^2 = 3^2 + 2^2 = 13 \qquad BC = \sqrt{13} \approx 3.61$$

$$AC^2 = CD^2 + DA^2 = 2^2 + 5^2 = 29 \qquad AC = \sqrt{29} \approx 5.39$$

Find the triangle's semi-perimeter.

$$s = \frac{AB + BC + AC}{2} = \frac{4.47 + 3.61 + 5.39}{2} \approx 6.74$$

Use Heron's formula.

$$A_{ABC} = \sqrt{s(s - AB)(s - BC)(s - AC)}$$

$$A = \sqrt{6.74(6.74 - 4.47)(6.74 - 3.61)(6.74 - 5.39)}$$

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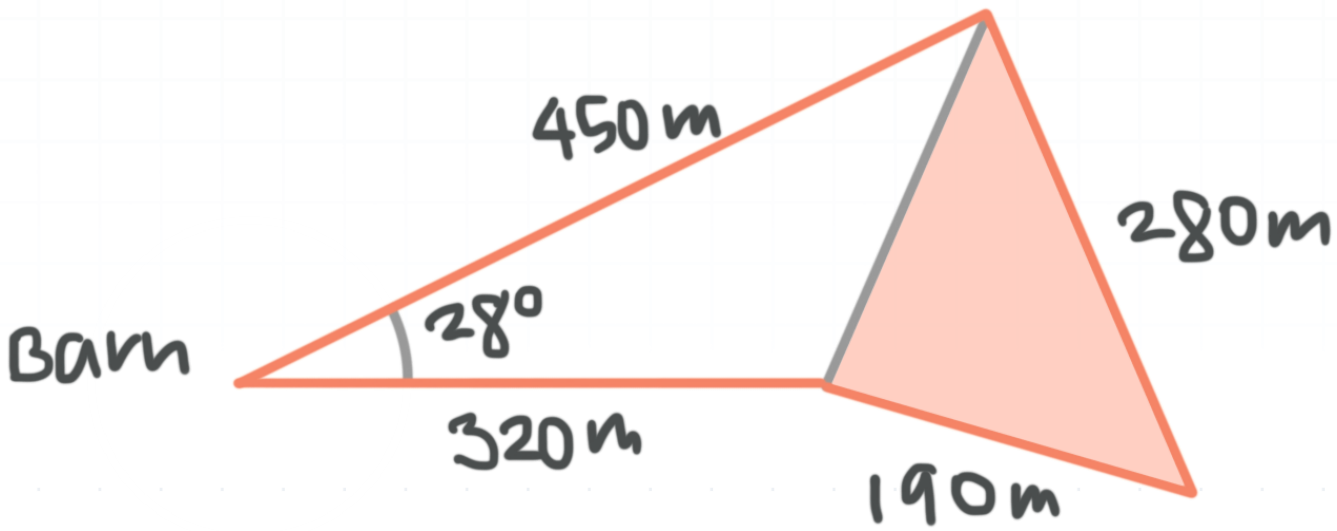
$A \approx 8$

■ 3. A plot of land is in the shape of an irregular quadrilateral. A barn is built on its westernmost vertex. The two sides of the plot adjacent to the barn have lengths 450 m and 320 m, with an included angle of $28°$. The other two sides are 280 m and 190 m.

A land owner decided to sell a triangular part of the land, formed by the two sides distant from the barn (shaded in the figure below). Help the land owner find the area of the plot he's going to sell.



Solution:

Label the quadrilateral as $ABCD$ with $AB = 450$ m, $BC = 280$ m, $CD = 190$ m, $AD = 320$ m, and $m\angle A = 28°$.

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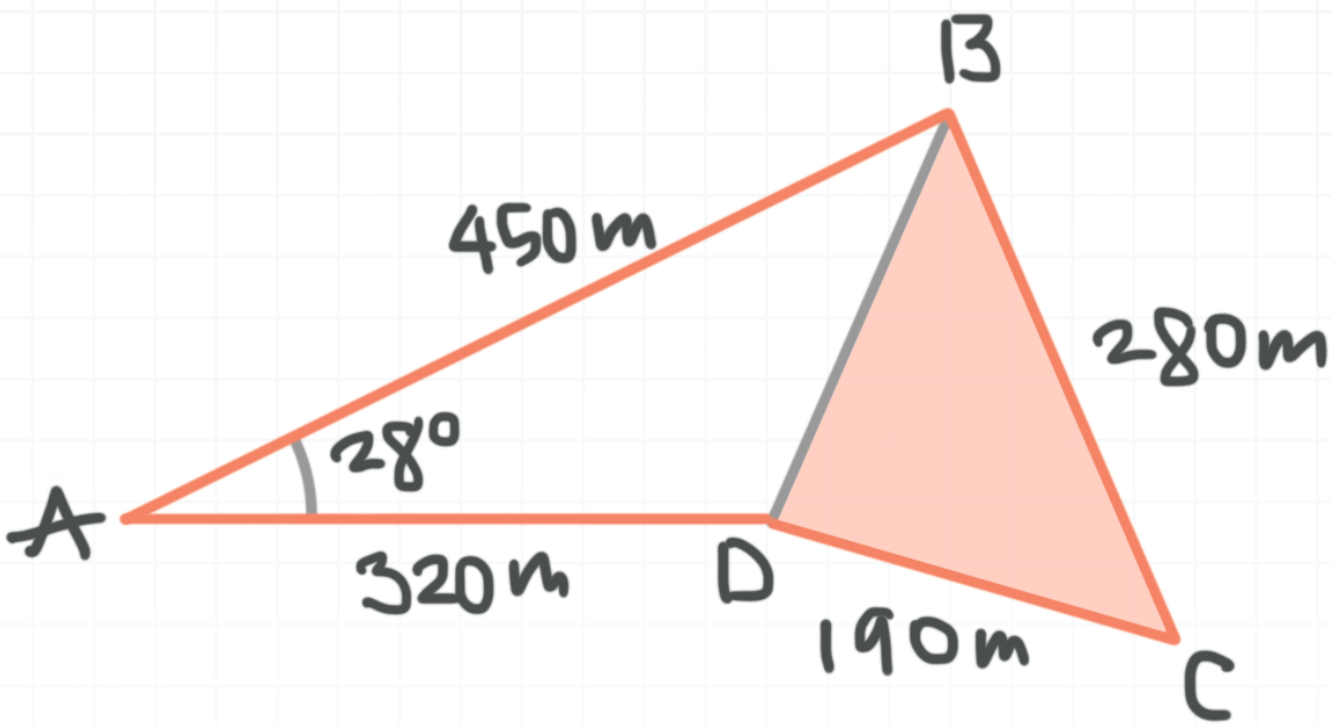

We need to evaluate the area of the triangle BCD, but we only know the lengths of two its sides, BC and CD. So in order to apply Heron's formula, we have to first find the length of the third side, BD.

We can find BD from triangle ABD using the law of cosines because we know the lengths of two adjacent sides, AB and AD, and we also know the angle between them, A. So by the law of cosines,

$$BD^2 = AB^2 + AD^2 - 2 \cdot AB \cdot AD \cdot \cos A$$

$$BD^2 = 450^2 + 320^2 - 2 \cdot 450 \cdot 320 \cdot \cos 28^\circ \approx 50{,}611$$

$$BD \approx \sqrt{50{,}611} \approx 225 \text{ m}$$

Now, with all three side lengths known, we can apply Heron's formula to find the area of BCD. Find its semi-perimeter.

$$s = \frac{BC + CD + BD}{2} = \frac{280 + 190 + 225}{2} \approx 348 \text{ m}$$

Apply Heron's formula.

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$$A_{BCD} = \sqrt{s(s - BC)(s - CD)(s - BD)}$$

$$A_{BCD} = \sqrt{348(348 - 280)(348 - 190)(348 - 225)}$$

$$A_{BCD} \approx 21{,}445 \text{ m}^2$$

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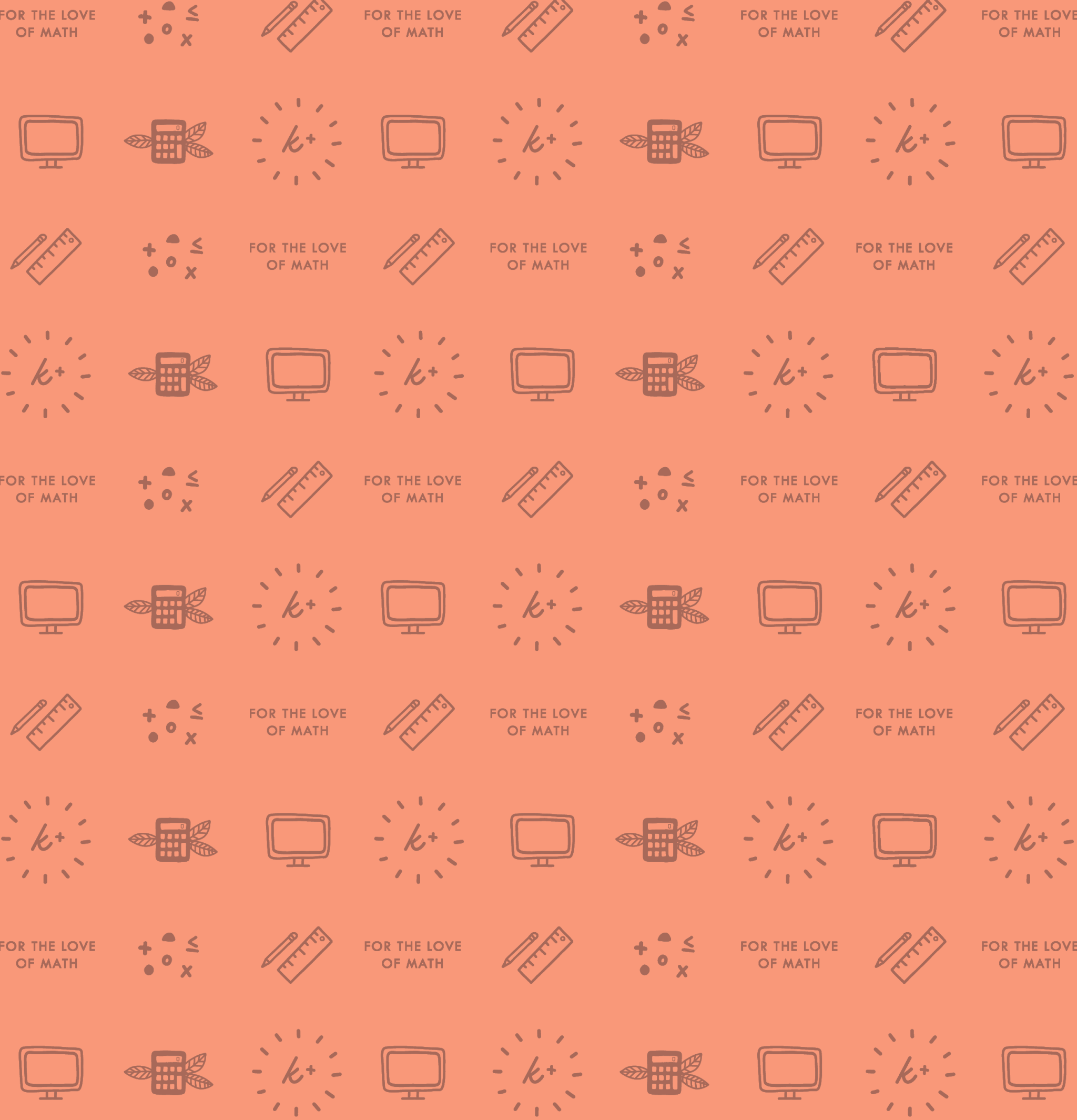
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